Automatic Control (1)

By

Associate Prof. / Mohamed Ahmed Ebrahim Mohamed

E-mail: mohamedahmed_en@yahoo.com

mohamed.mohamed@feng.bu.edu.eg

Web site: http://bu.edu.eg/staff/mohamedmohamed033

Lecture (3)

Block Diagram Reduction using Decomposition

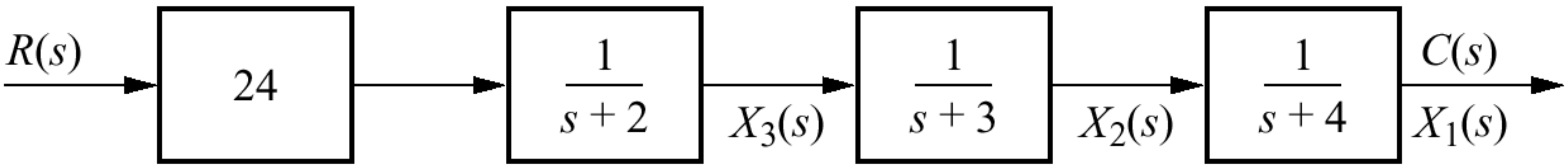
&

Time Domain Analysis

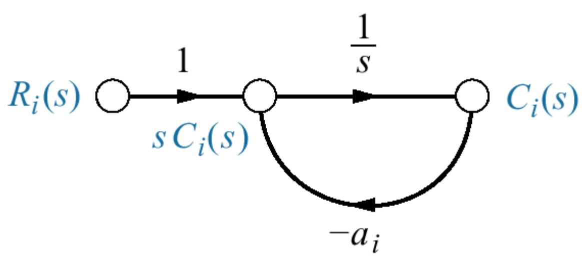
Decomposition

Associate Prof. Dr . Mohamed Ahmed Ebrahim

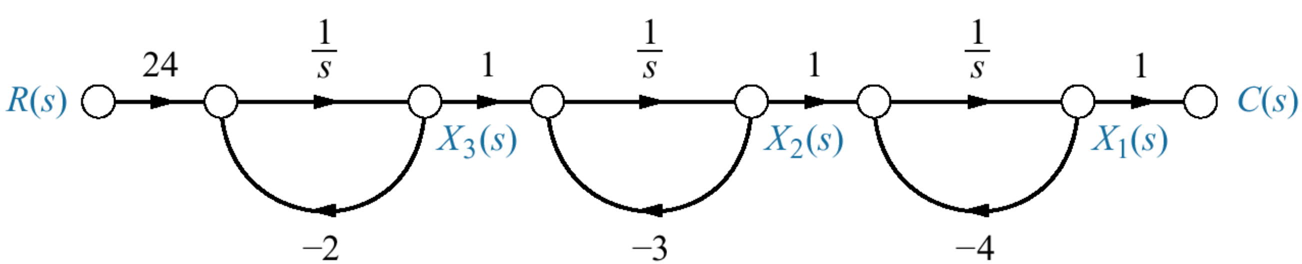
Alternate Representation: Cascade Form



$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$



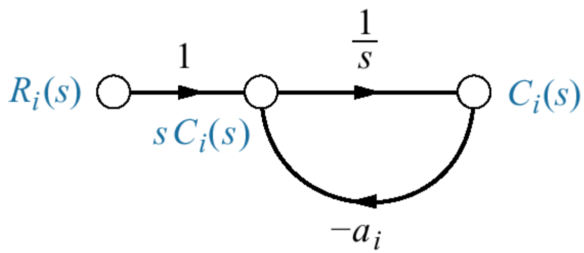
(*a*)



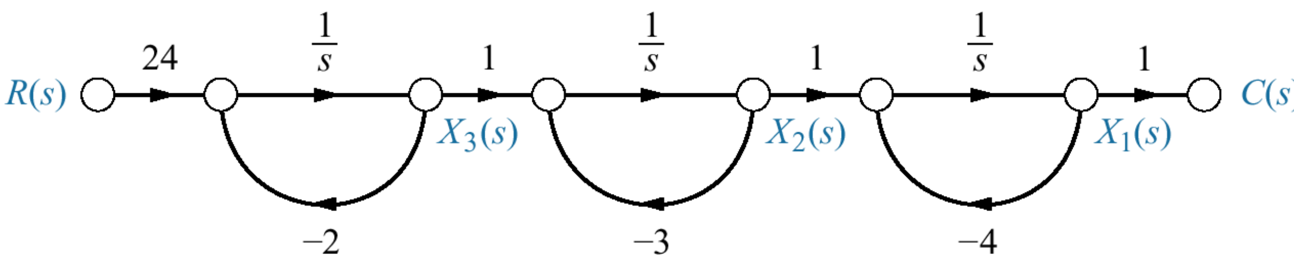
(*b*)

Ebrahim

Alternate Representation: Cascade Form



(*a*)



(*b*)

$$\dot{x}_1 = -4x_1 + x_2$$
$$\dot{x}_2 = -3x_2 + x_3$$
$$\dot{x}_3 = -2x_3 + 24r$$
$$y = c(t) = x_1$$

$$\dot{X} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

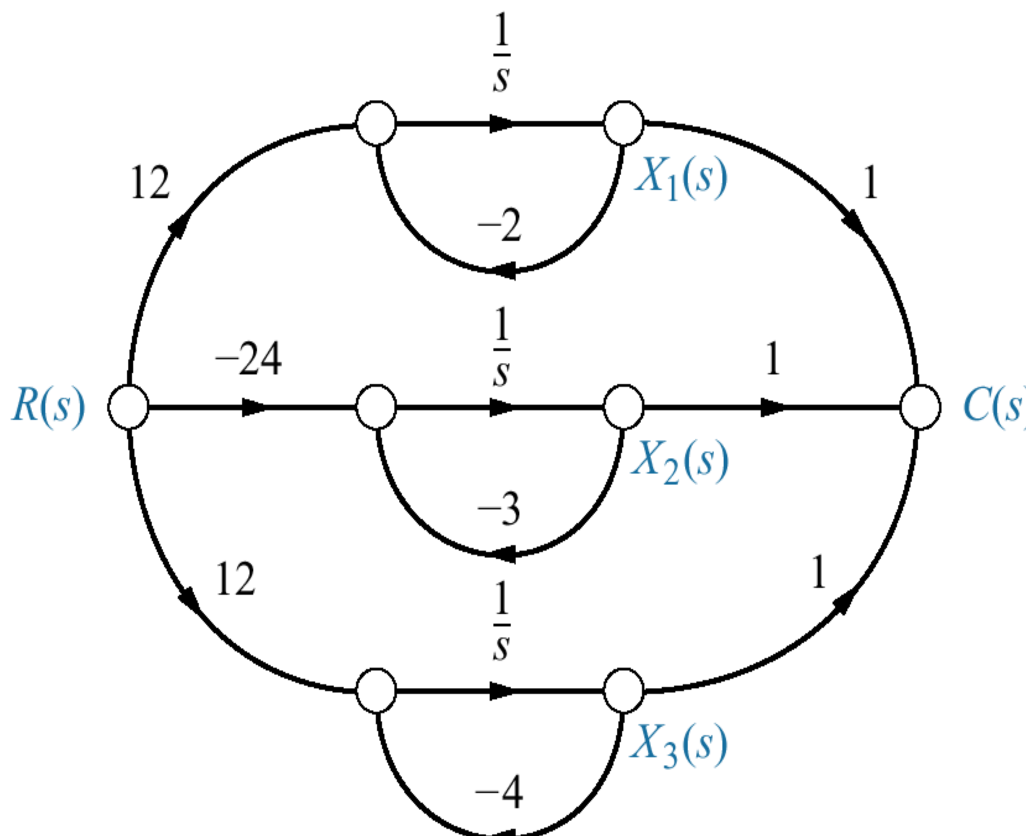
Alternate Representation: Parallel Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

$$\dot{x}_1 = -2x_1 \qquad\qquad\qquad\qquad + 12r$$
$$\dot{x}_2 = \qquad\qquad -3x_2 \qquad\qquad -24r$$
$$\dot{x}_3 = \qquad\qquad\qquad\qquad -4x_3 + 12r$$
$$y = c(t) = x_1 + x_2 + x_3$$

$$\dot{X} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} X + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X$$



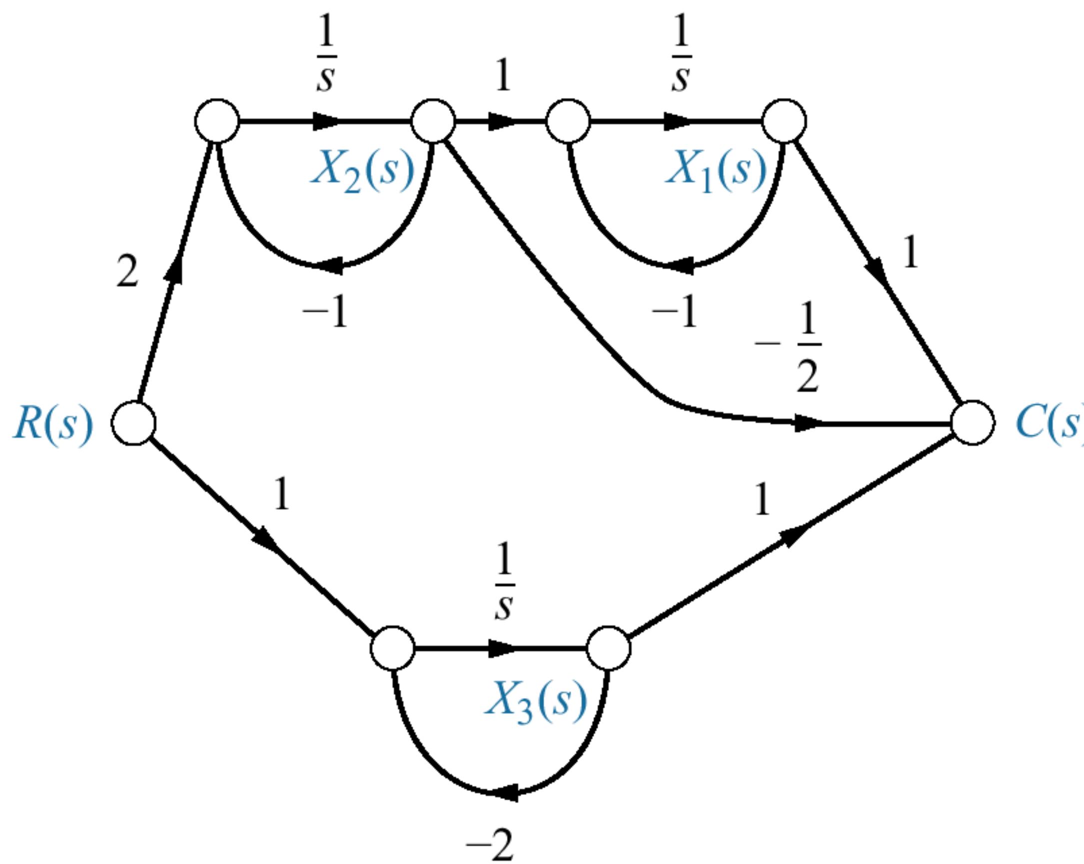
Alternate Representation: Parallel Form Repeated roots

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = x_2 + 2r$$
$$\dot{x}_3 = -2x_3 + r$$
$$y = c(t) = x_1 - 1/2x_2 + x_3$$

$$\dot{X} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & -1/2 & 1 \end{bmatrix} X$$

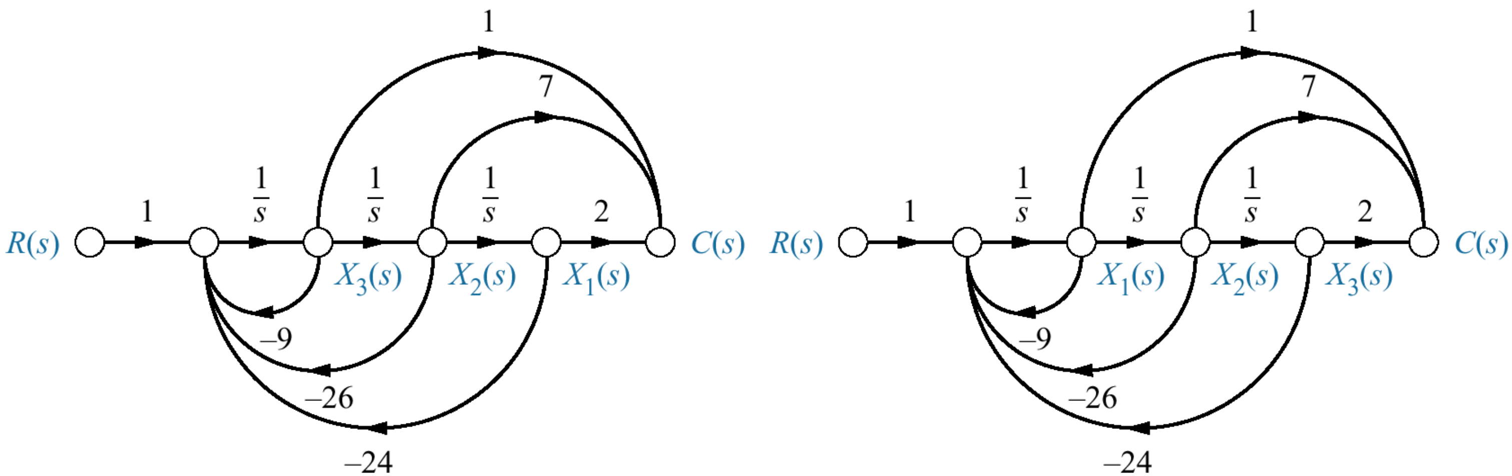


Associate Prof. Dr. Mohamed Ahmed Ebrahim

Alternate Representation: controller canonical form

G(s) = C(s)/R(s) = $(s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

This form is obtained from the phase-variable form simply by ordering the phase variable in reverse order



(a) (b)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix} r$$

$$y = \begin{bmatrix}2 & 7 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-9 & -26 & -24\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix} r$$

$$y = \begin{bmatrix}1 & 7 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Alternate Representation: controller canonical form

System matrices that contain the coefficients of the characteristic polynomial are called *companion matrices* to the characteristic polynomial.

Phase-variable form result in lower companion matrix

Controller canonical form results in upper companion matrix

Alternate Representation: observer canonical form

Observer canonical form so named for its use in the design of observers

$G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

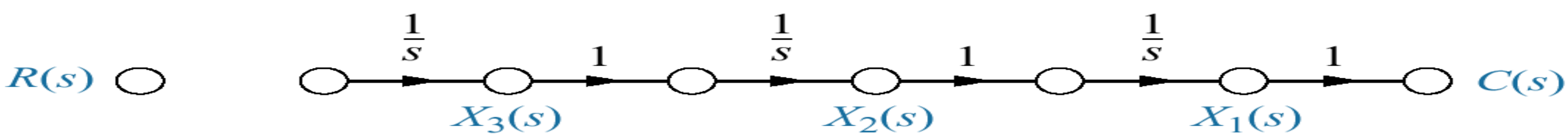
$= (1/s+7/s^2 +2/s^3)/(1+9/s+26/s^2 +24/s^3)$

Cross multiplying

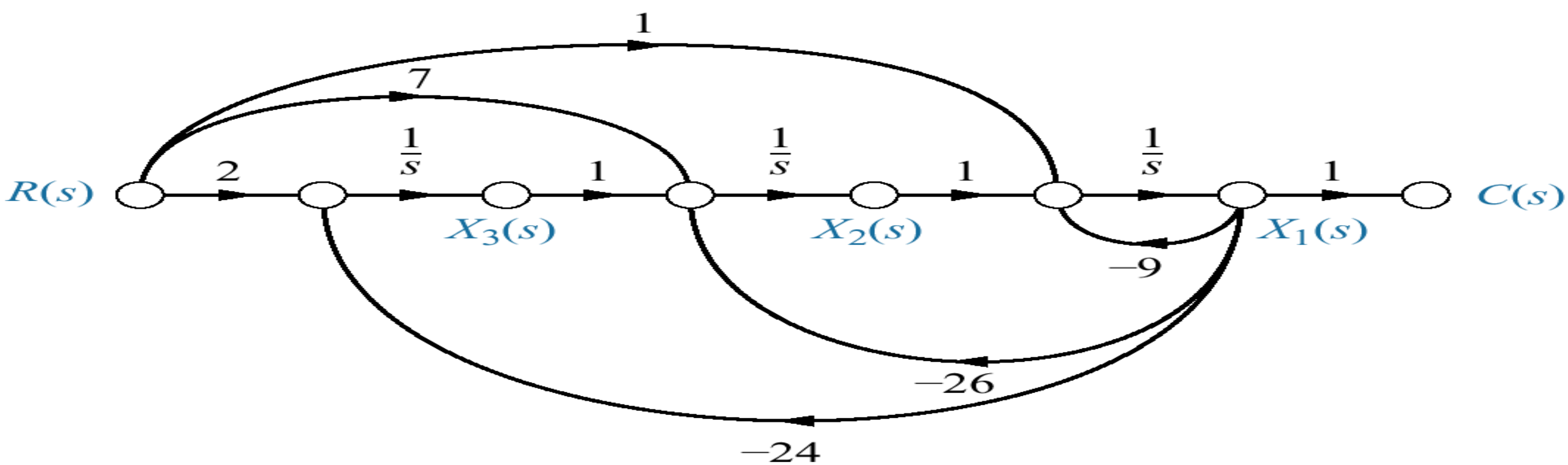
$(1/s+7/s^2 +2/s^3)R(s) = (1+9/s+26/s^2 +24/s^3)\ C(s)$

And $C(s) = 1/s[R(s)-9C(s)] +1/s^2[7R(s)-26C(s)]+1/s^3[2R(s)-24C(s)]$

$= 1/s\{ [R(s)-9C(s)] + 1/s \{[7R(s)-26C(s)]+1/s\ [2R(s)-24C(s)]\}\}$



(a)



(b)

Alternate Representation: observer canonical form

$$\dot{x}_1 = -9x_1 + x_2 + r$$

$$\dot{x}_2 = -26x_1 + x_3 + 7r$$

$$\dot{x}_3 = -24x_1 + 2r$$

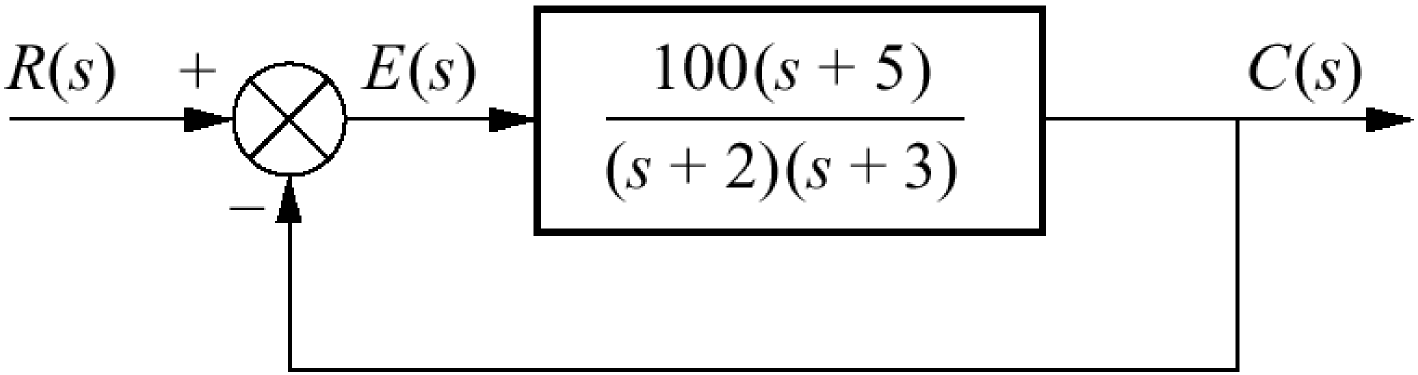
$$y = c(t) = x_1$$

$$\dot{X} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

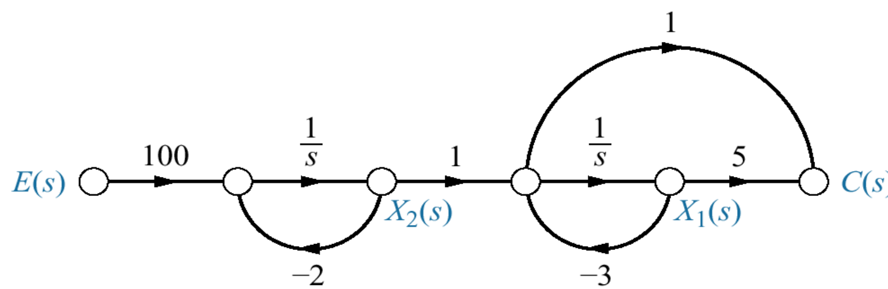
Note that the observer form has A matrix that is transpose of the controller canonical form, B vector is the transpose of the controller C vector, and C vector is the transpose of the controller B vector. The 2 forms are called duals.

Feedback Control System Example



Problem Represent the feedback control system shown in state space. Model the forward transfer function in cascade form.

Solution first we model the forward transfer function as in (a), Second we add the feedback and input paths as shown in (b) complete system. Write state equations

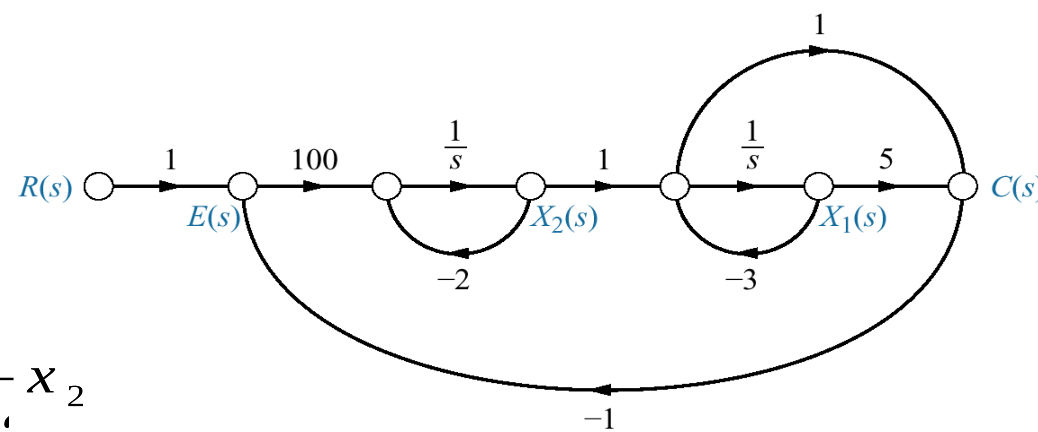


(a)

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -2x_2 + 100(r - c)$$

$$\text{but } c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2$$



(b)

Feedback Control System Example

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -200x_1 - 102x_2 + 100r$$

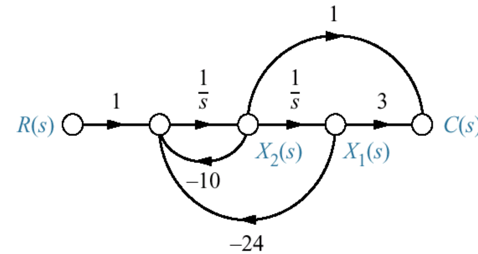
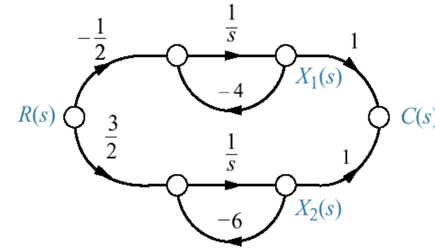
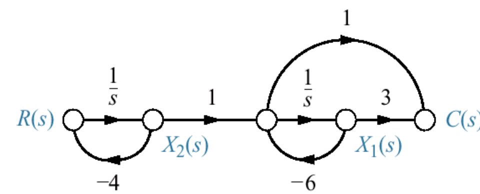
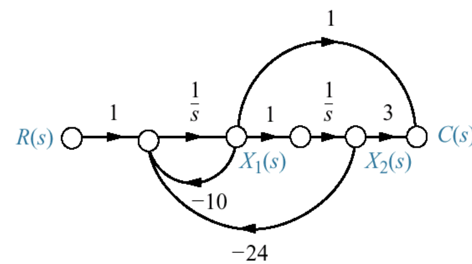
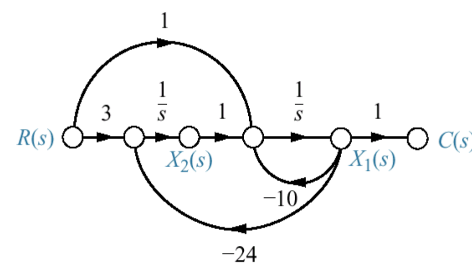
$$y = c(t) = 2x_1 + x_2$$

$$\dot{X} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} X + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} X$$

State-space forms for

C(s)/R(s) =(s+ 3)/[(s+ 4)(s+ 6)].
Note: y = c(t)

Form	Transfer Function	Signal-Flow Diagram	State Equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \quad 1]\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$		$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \quad 1]\mathbf{x}$
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$		$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \quad 1]\mathbf{x}$
Controller canonical	$\frac{1}{(s^2+10s+24)} * (s+3)$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$ $y = [1 \quad 3]\mathbf{x}$
Observer canonical	$\frac{\frac{1}{s} + \frac{3}{s^2}}{1 + \frac{10}{s} + \frac{24}{s^2}}$		$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$ $y = [1 \quad 0]\mathbf{x}$

Time Domain Analysis

Associate Prof. Dr . Mohamed Ahmed Ebrahim

Introduction

- In time-domain analysis the response of a dynamic system to an input is expressed as a function of time.
- It is possible to compute the time response of a system if the nature of input and the mathematical model of the system are known.
- Usually, the input signals to control systems are not known fully ahead of time.
- It is therefore difficult to express the actual input signals mathematically by simple equations.

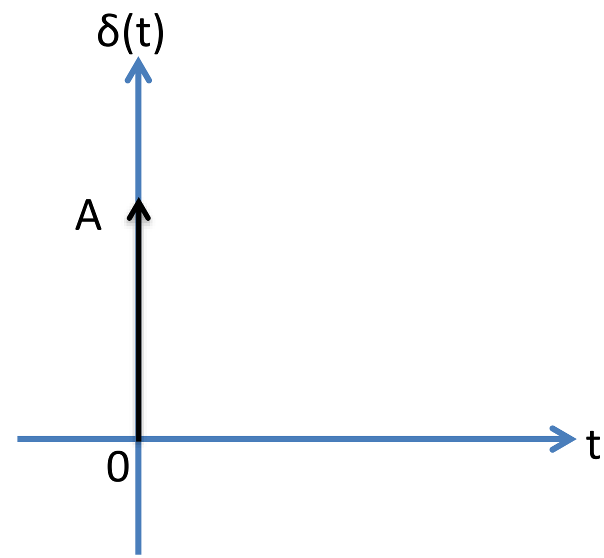
Standard Test Signals

- The characteristics of actual input signals are a sudden shock, a sudden change, a constant velocity, and constant acceleration.
- The dynamic behavior of a system is therefore judged and compared under application of standard test signals – an impulse, a step, a constant velocity, and constant acceleration.
- The other standard signal of great importance is a sinusoidal signal.

Standard Test Signals

- Impulse signal
  - The impulse signal imitate the sudden shock characteristic of actual input signal.

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

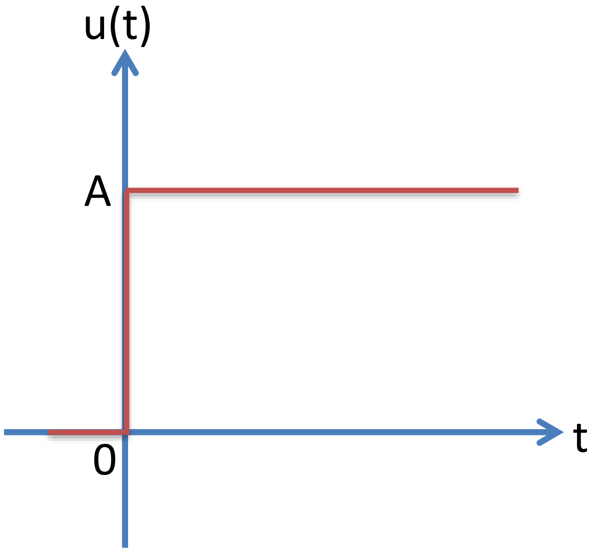


  - If A=1, the impulse signal is called unit impulse signal.

Standard Test Signals

- Step signal
  - The step signal imitate the sudden change characteristic of actual input signal.

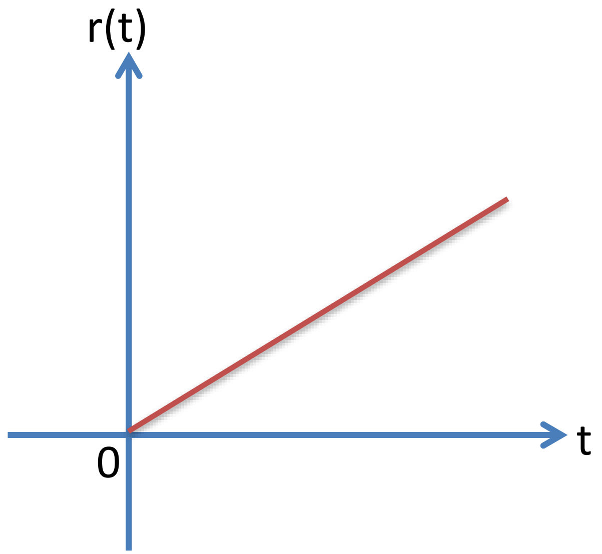
$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$



  - If A=1, the step signal is called unit step signal

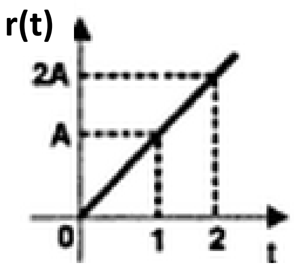
Standard Test Signals

- Ramp signal
  - The ramp signal imitate the constant velocity characteristic of actual input signal.



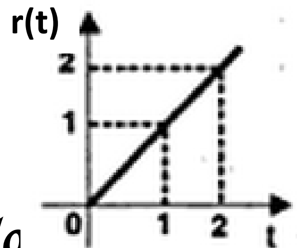
$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

ramp signal with slope A



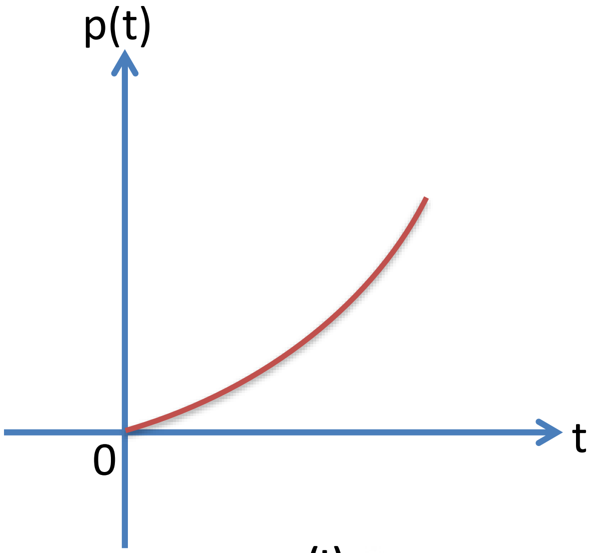
  - If *A=1*, the ramp signal is called unit ramp signal

unit ramp signal

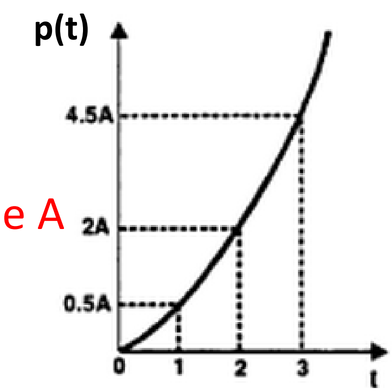


Standard Test Signals

- Parabolic signal
  - The parabolic signal imitate the constant acceleration characteristic of actual input signal.

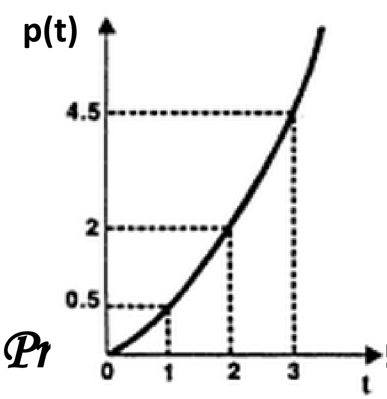


$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$



parabolic signal with slope A

  - If *A=1*, the parabolic signal is called unit parabolic signal.



Unit parabolic signal

Relation between standard Test Signals

- Impulse
- Step
- Ramp
- Parabolic

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$\int$ (downward) / $\frac{d}{dt}$ (upward)

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ (downward) / $\frac{d}{dt}$ (upward)

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$\int$ (downward) / $\frac{d}{dt}$ (upward)

$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

Laplace Transform of Test Signals

- Impulse

$$\delta(t) = \begin{cases} A & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$L\{\delta(t)\} = \delta(s) = A$$

- Step

$$u(t) = \begin{cases} A & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{u(t)\} = U(s) = \frac{A}{S}$$

Associate Prof. Dr . Mohamed Ahmed Ebrahim

Laplace Transform of Test Signals

- Ramp

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{r(t)\} = R(s) = \frac{A}{s^2}$$

- Parabolic

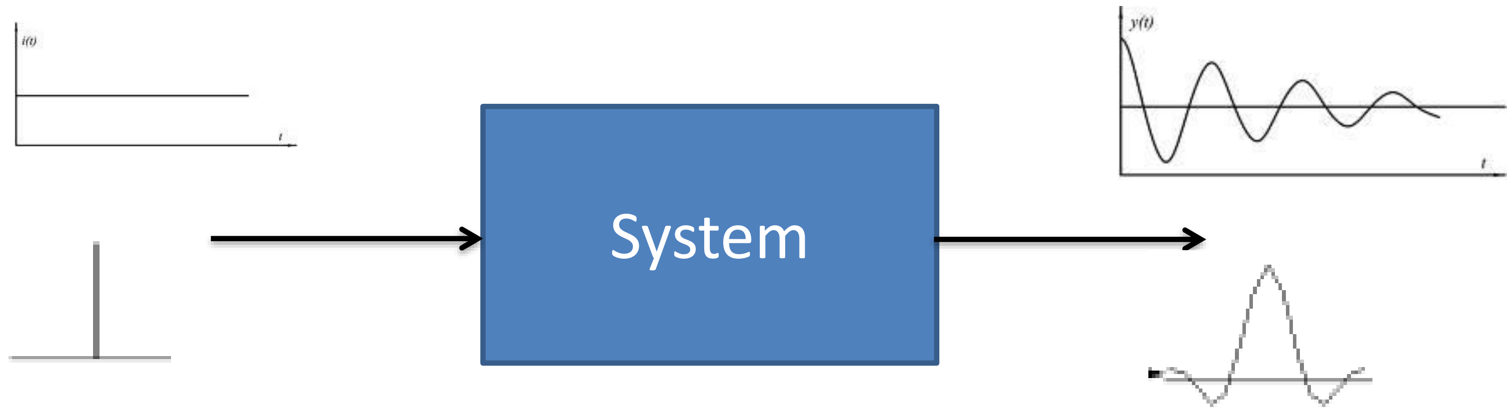
$$p(t) = \begin{cases} \dfrac{At^2}{2} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$L\{p(t)\} = P(s) = \frac{A}{S^3}$$

Associate Prof. Dr . Mohamed Ahmed Ebrahim

Time Response of Control Systems

- Time response of a dynamic system response to an input expressed as a function of time.



- The time response of any system has two components
  - Transient response
  - Steady-state response.

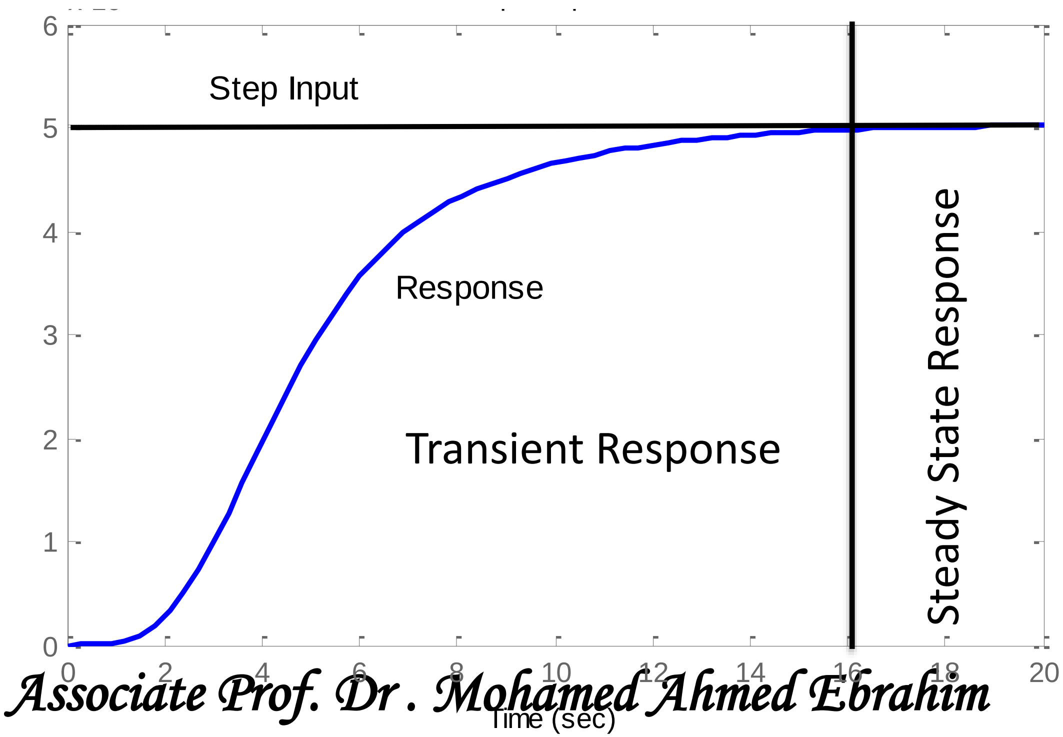
Associate Prof. Dr . Mohamed Ahmed Ebrahim

Time Response of Control Systems

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.



Associate Prof. Dr . Mohamed Ahmed Ebrahim

Time Response of Control Systems

- Transient response depend upon the system poles only and not on the type of input.
- It is therefore sufficient to analyze the transient response using a step input.
- The steady-state response depends on system dynamics and the input quantity.
- It is then examined using different test signals by final value theorem.

Introduction

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

- Where ***K*** is the D.C gain and ***T*** is the time constant of the system.
- Time constant is a measure of how quickly a 1st order system responds to a unit step input.
- D.C Gain of the system is ratio between the input signal and the steady state value of output.

Introduction

- The first order system given below.

$$G(s) = \frac{10}{5s + 1}$$

- D.C gain is 10 and time constant is 5 seconds.

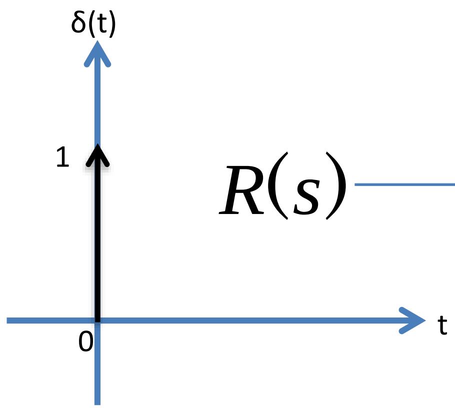
- For the following system

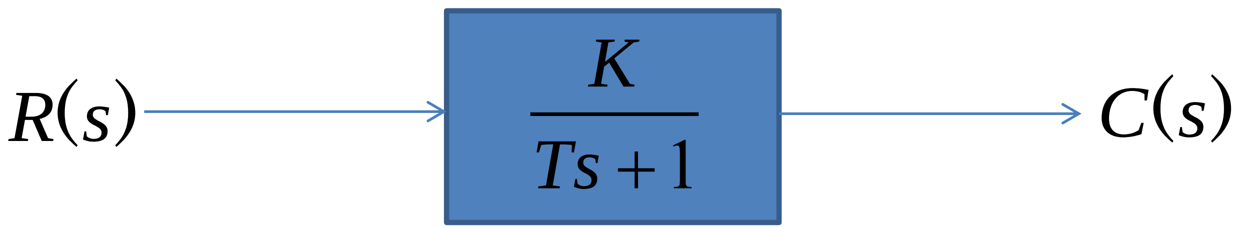
$$G(s) = \frac{6}{s + 2} \qquad = \frac{6/2}{1/2s + 1}$$

- D.C Gain of the system is 6/2 and time constant is 1/2 seconds.

Impulse Response of 1st Order System

- Consider the following 1st order system





$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

Impulse Response of 1st Order System

$$C(s) = \frac{K}{Ts+1}$$

- Re-arrange following equation as

$$C(s) = \frac{K/T}{s+1/T}$$

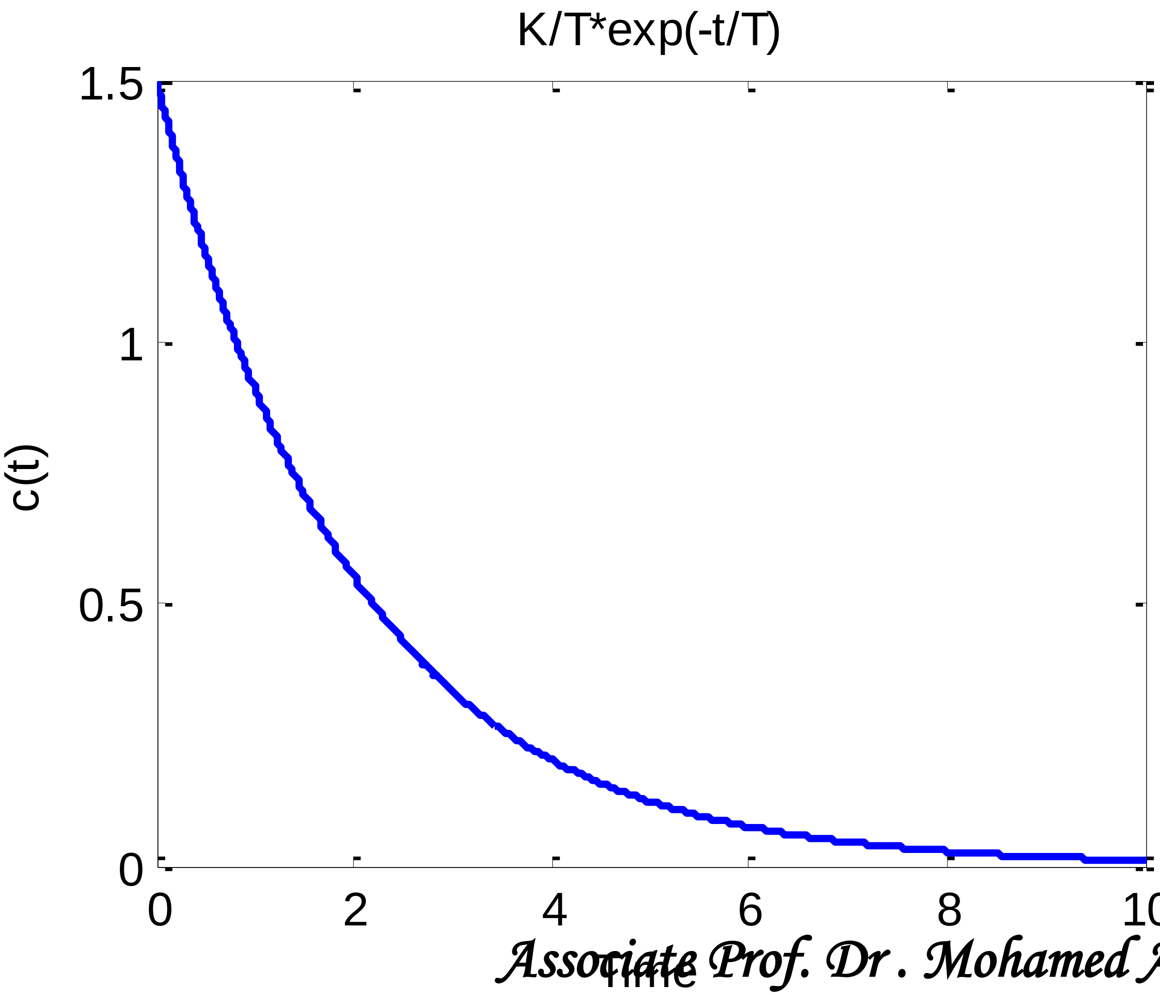
- In order to compute the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$L^{-1}\left(\frac{C}{s+a}\right) = Ce^{-at} \qquad c(t) = \frac{K}{T}e^{-t/T}$$

Associate Prof. Dr . Mohamed Ahmed Ebrahim

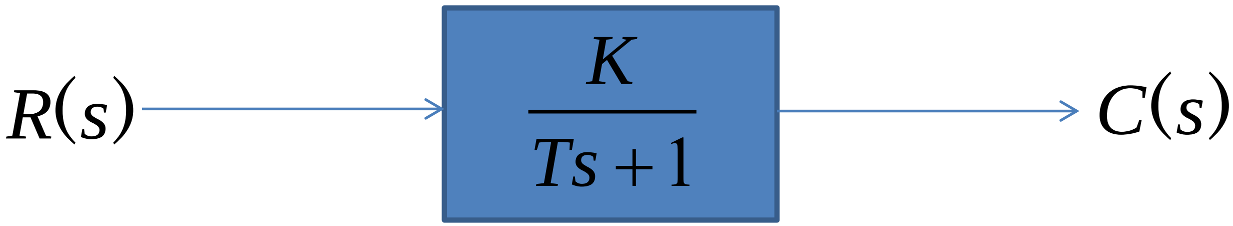
Impulse Response of 1st Order System

- If K=3 and T=2s then $c(t)=\frac{K}{T}e^{-t/T}$



Step Response of 1st Order System

- Consider the following 1st order system



$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion.

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

Step Response of 1st Order System

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

$$c(t) = K\left(1 - e^{-t/T}\right)$$

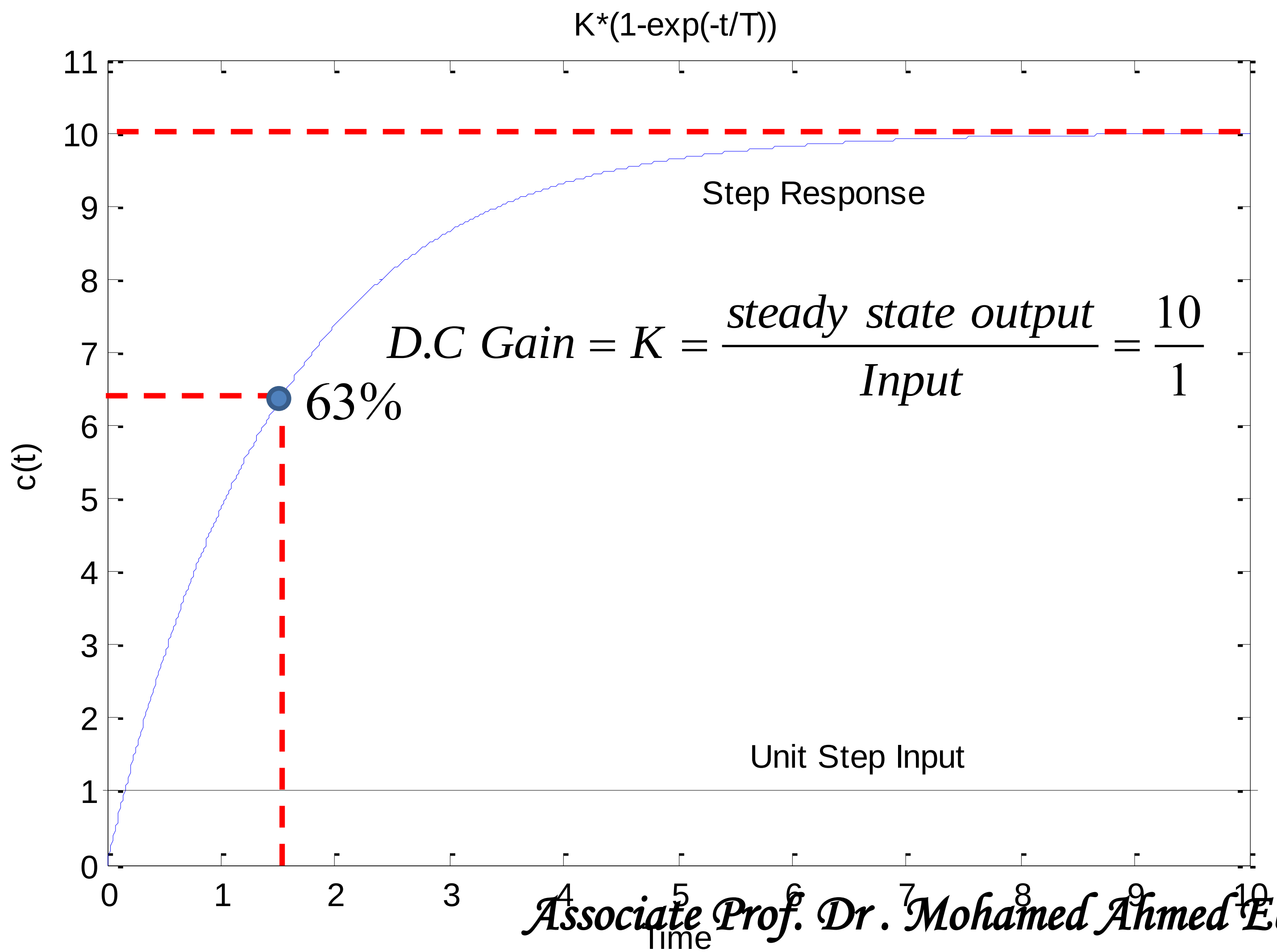
- When t=T (time constant)

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

Step Response of 1st Order System

- If K=10 and T=1.5s then $$c(t) = K\left(1 - e^{-t/T}\right)$$

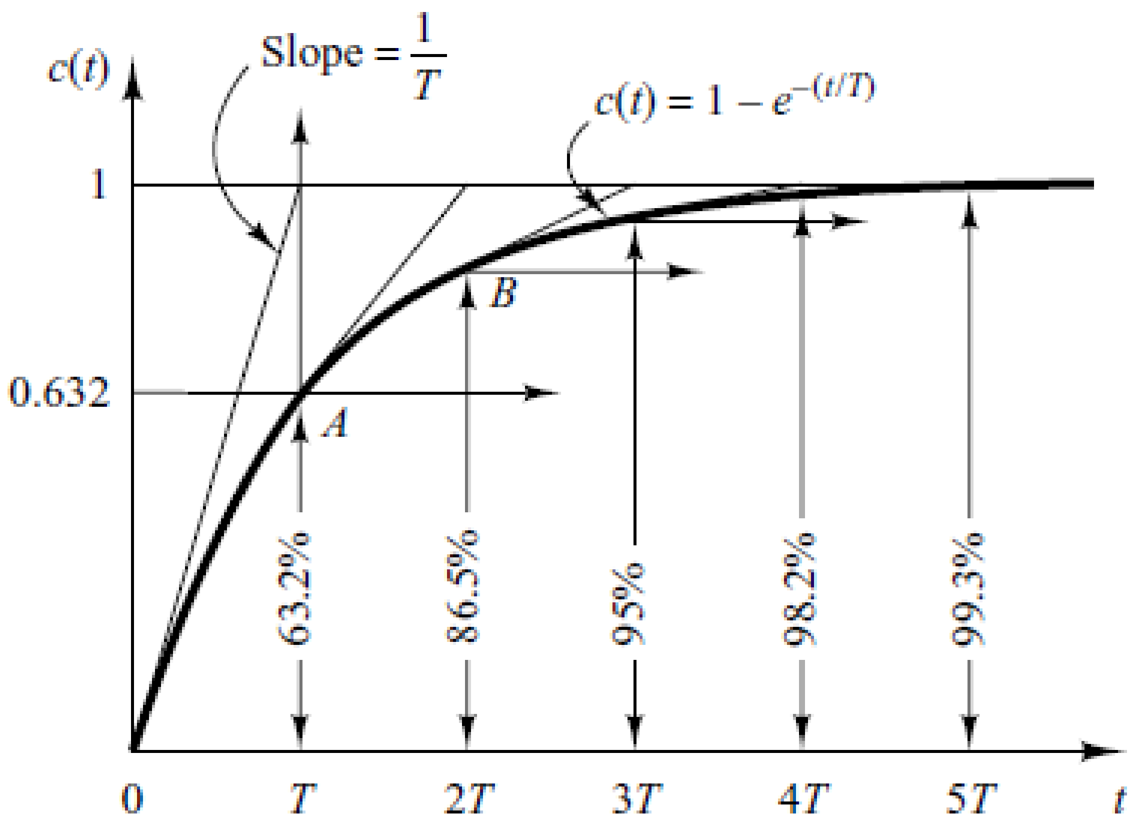


$$D.C\ Gain = K = \frac{steady\ \ state\ \ output}{Input} = \frac{10}{1}$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

Step Response of 1st order System

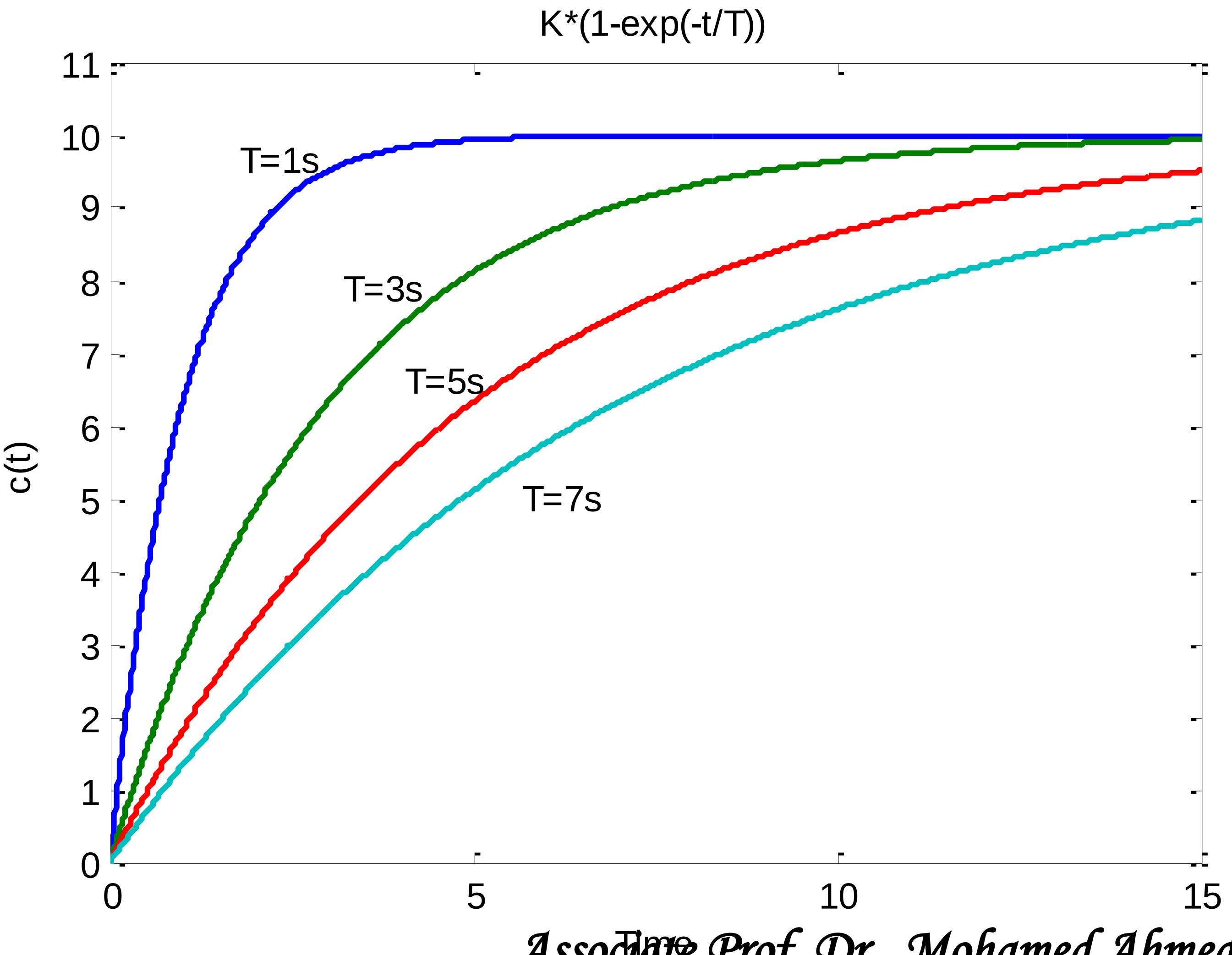
- System takes five time constants to reach its final value.



Ahmed Ebrahim

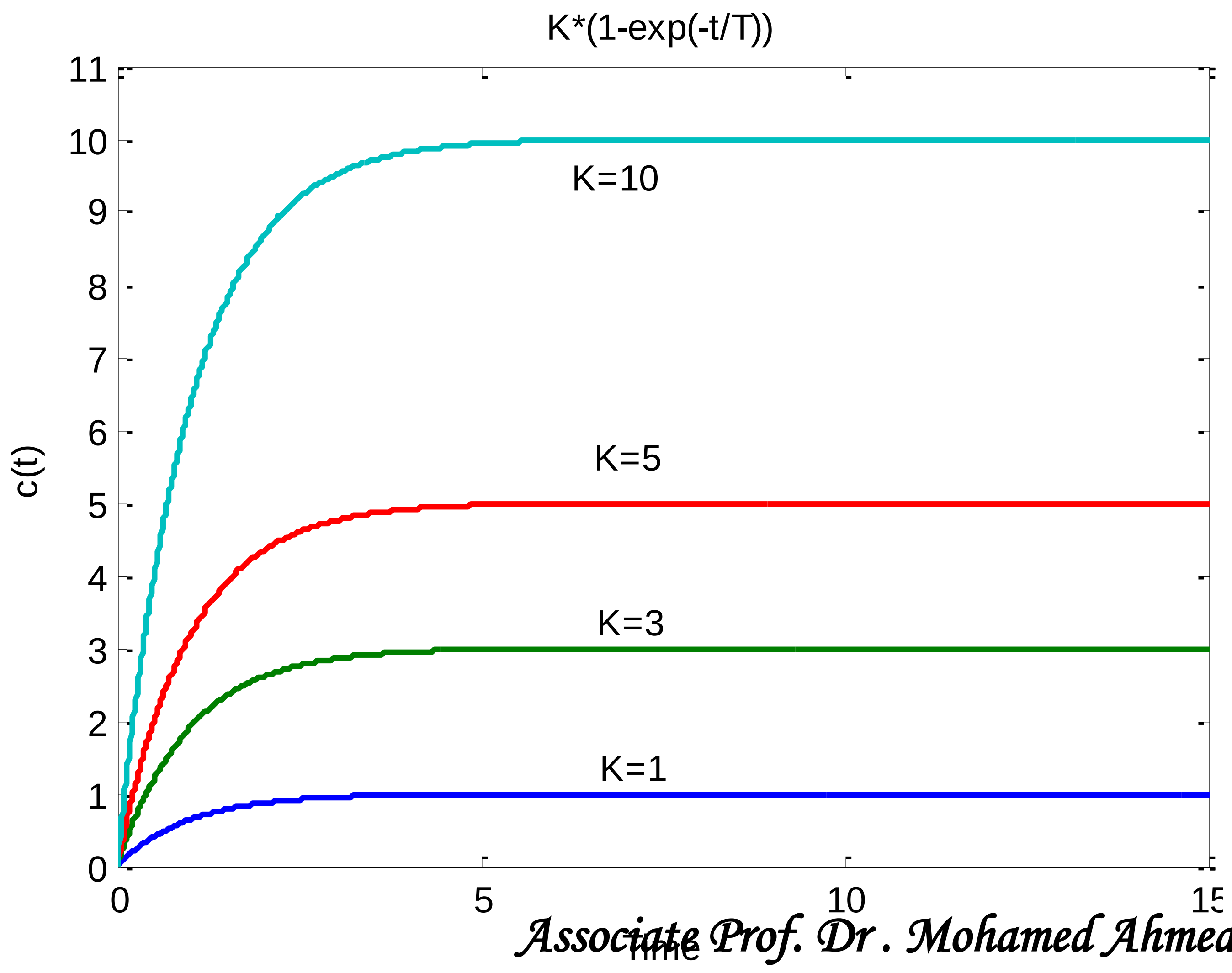
Step Response of 1st Order System

- If K=10 and T=1, 3, 5, 7 $c(t) = K\left(1 - e^{-t/T}\right)$



Step Response of 1st Order System

- If K=1, 3, 5, 10 and T=1 $c(t) = K\left(1 - e^{-t/T}\right)$



Associate Prof. Dr . Mohamed Ahmed Ebrahim

Relation Between Step and impulse response

- The step response of the first order system is

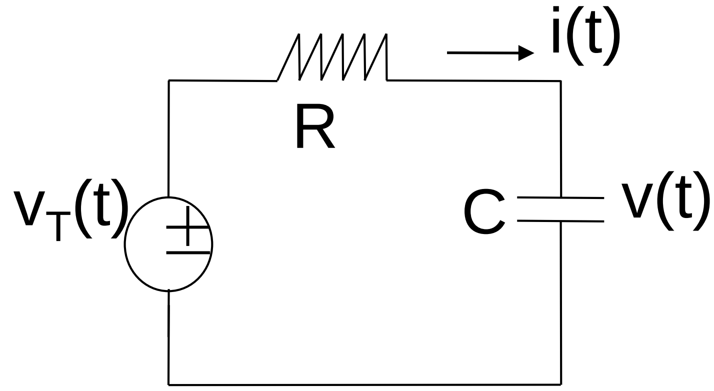
$$c(t) = K\left(1 - e^{-t/T}\right) = K - Ke^{-t/T}$$

- Differentiating c(t) with respect to t yields

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

$$\frac{dc(t)}{dt} = \frac{K}{T}e^{-t/T}$$

Analysis of Simple RC Circuit



$$R \cdot i(t) + v(t) = v_T(t)$$

$$i(t) = \frac{d(Cv(t))}{dt} = C\frac{dv(t)}{dt}$$

$$\Rightarrow RC\frac{dv(t)}{dt} + v(t) = v_T(t)$$

state variable — $v(t)$

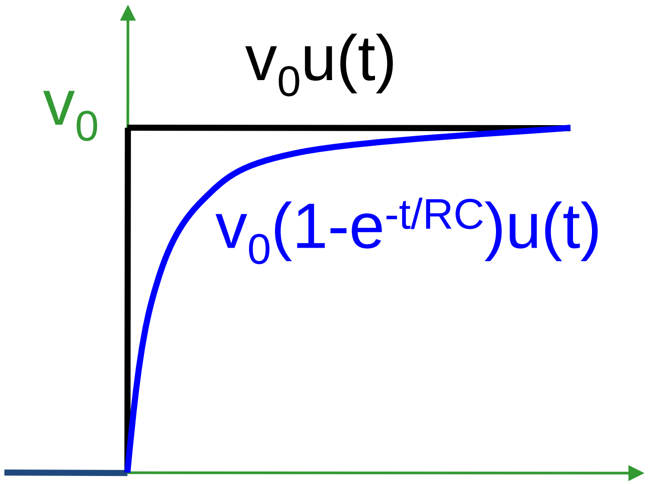
Input waveform — $v_T(t)$

Analysis of Simple RC Circuit

Step-input response:

$$RC\frac{dv(t)}{dt}+v(t)=v_0u(t)$$

$$v(t)=Ke^{-t/RC}+v_0u(t)$$



match initial state:

$$v(0)=0 \quad \Rightarrow K+v_0u(t)=0 \Rightarrow K+v_0=0$$

output response for step-input:

$$v(t)=v_0(1-e^{-t/RC})u(t)$$

Example 1

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T
  - D.C Gain K
  - Transfer Function
  - Step Response

Example 1

- The Laplace Transform of Impulse response of a system is actually the transfer function of the system.
- Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

Example 1

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T=2
  - D.C Gain K=6
  - Transfer Function $\frac{C(s)}{R(s)} = \frac{6}{2S+1}$
  - Step Response

Example 1

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)dt = 3\int e^{-0.5t}dt$$

$$c_s(t) = -6e^{-0.5t} + C$$

- We can find out C if initial condition is known e.g. $c_s(0)=0$

$$0 = -6e^{-0.5\times 0} + C$$
$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

Example 1

- If initial conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

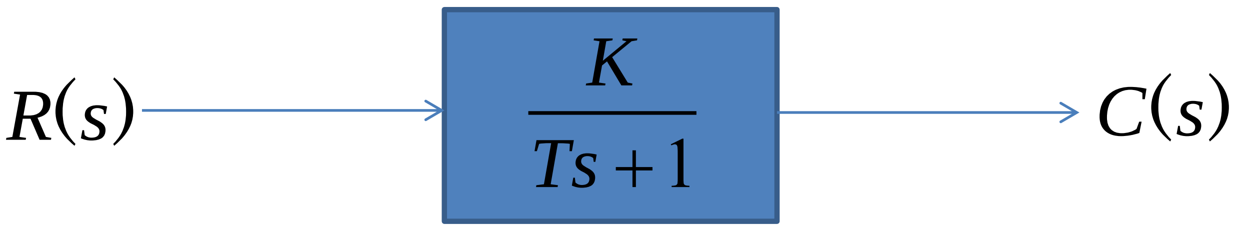
$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

Ramp Response of 1st Order System

- Consider the following 1st order system



$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{K}{s^2(Ts+1)}$$

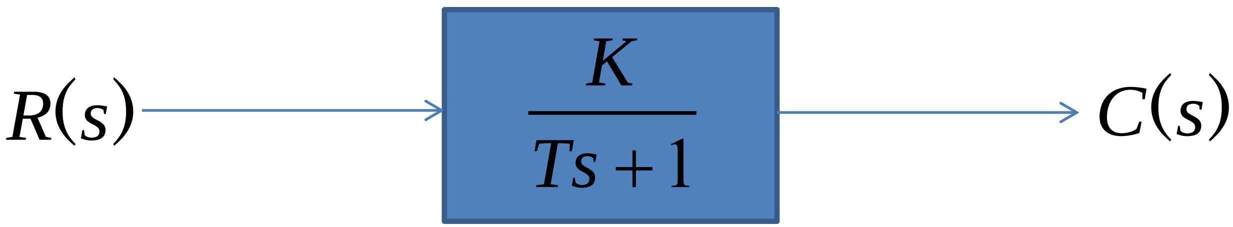
- The ramp response is given as

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$

Associate Prof. Dr . Mohamed Ahmed Ebrahim

Parabolic Response of 1st Order System

- Consider the following 1st order system



$$R(s) = \frac{1}{s^3} \quad \text{Therefore,} \quad C(s) = \frac{K}{s^3(Ts+1)}$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

Practical Determination of Transfer Function of 1^{st} Order Systems

- Often it is not possible or practical to obtain a system's transfer function analytically.
- Perhaps the system is closed, and the component parts are not easily identifiable.
- The system's step response can lead to a representation even though the inner construction is not known.
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.

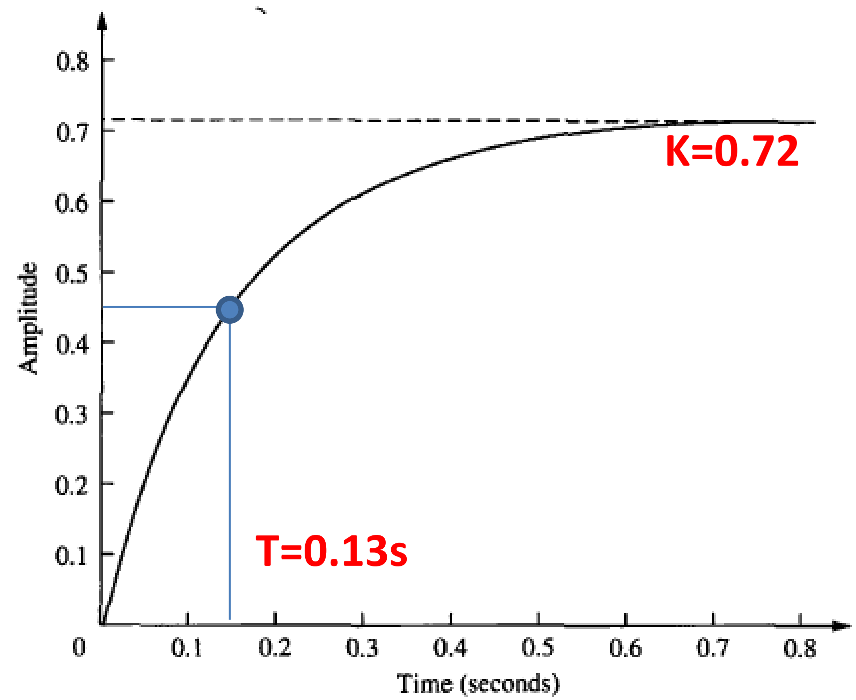
Practical Determination of Transfer Function of 1st Order Systems

- If we can identify *T* and *K* empirically we can obtain the transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts + 1}$$

Practical Determination of Transfer Function of 1st Order Systems

- For example, assume the unit step response given in figure.
- From the response, we can measure the time constant, that is, the time for the amplitude to reach 63% of its final value.
- Since the final value is about 0.72 the time constant is evaluated where the curve reaches 0.63 x 0.72 = 0.45, or about 0.13 second.
- K is simply steady state value.



- Thus transfer function is obtained as:

$$\frac{C(s)}{R(s)} = \frac{0.72}{0.13s+1} = \frac{5.5}{s+7.7}$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

First Order System with a Zero

$$\frac{C(s)}{R(s)} = \frac{K(1+\alpha s)}{Ts+1}$$

- Zero of the system lie at *-1/α* and pole at *-1/T*.

- Step response of the system would be:

$$C(s) = \frac{K(1+\alpha s)}{s(Ts+1)}$$

$$C(s) = \frac{K}{s} + \frac{K(\alpha - T)}{(Ts+1)}$$

$$c(t) = K + \frac{K}{T}(\alpha - T)e^{-t/T}$$

Associate Prof. Dr. Mohamed Ahmed Ebrahim

First Order System With Delays

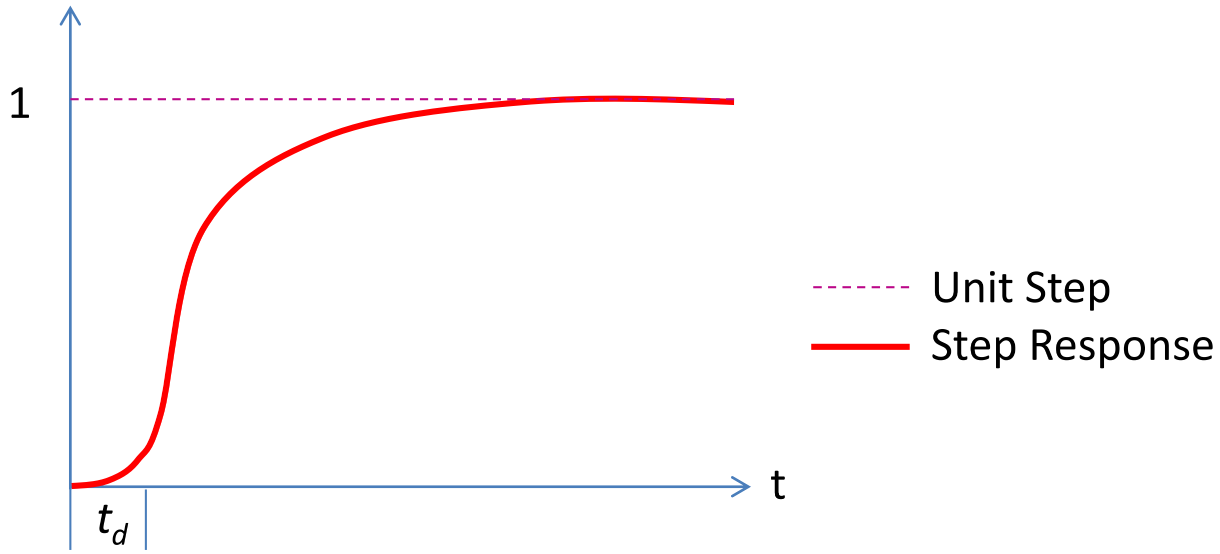
- Following transfer function is the generic representation of 1st order system with time lag.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}e^{-st_d}$$

- Where t_d is the delay time.

First Order System With Delays

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}e^{-st_d}$$



First Order System With Delays

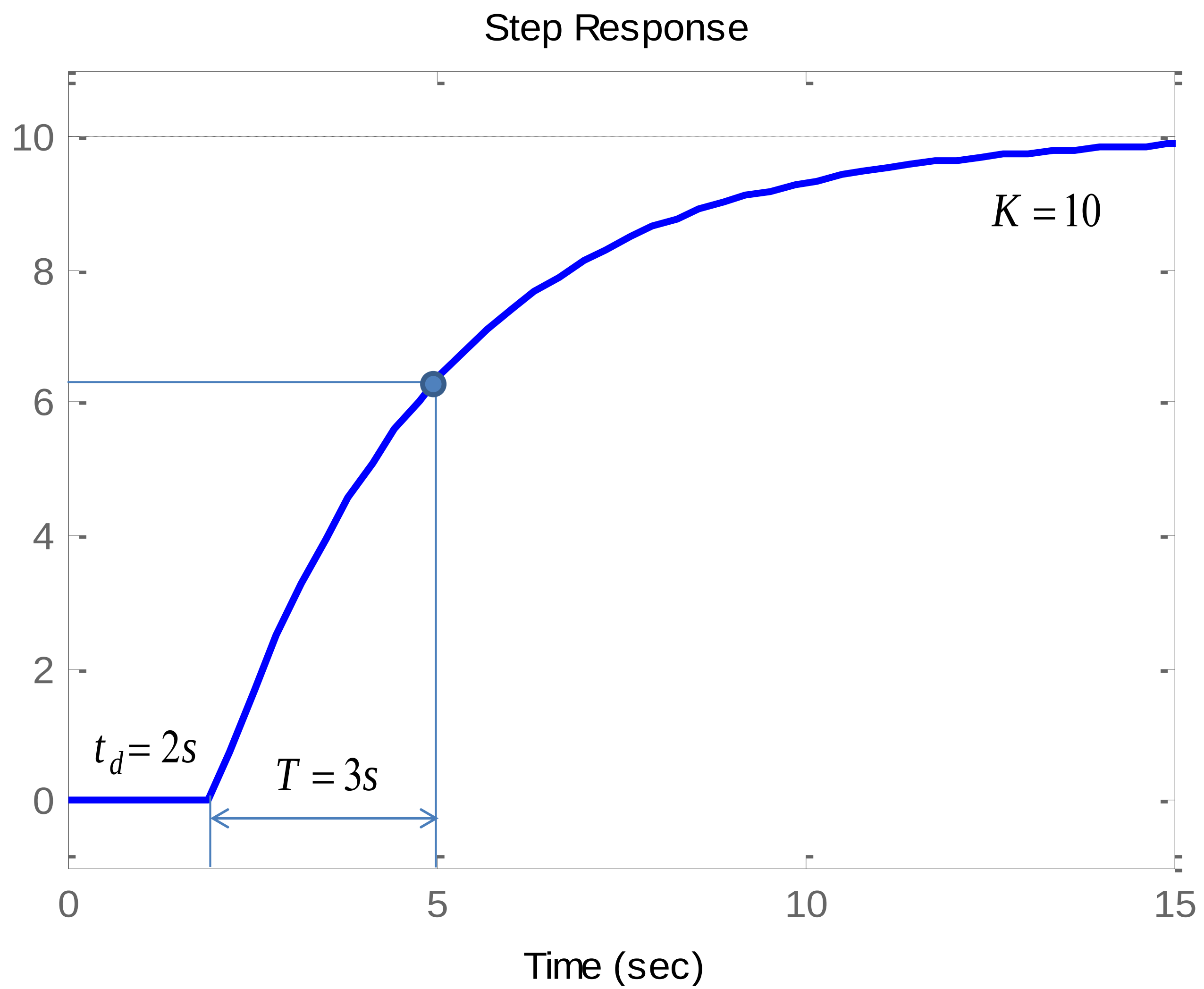
$$\frac{C(s)}{R(s)} = \frac{10}{3s+1}e^{-2s}$$

$$C(s) = \frac{10}{s(3s+1)}e^{-2s}$$

$$L^{-1}[e^{-\partial s}F(s)] = f(t-\partial)u(t-\partial)$$

$$L^{-1}[(\frac{10}{s} + \frac{-10}{s+1/3})e^{-2s}] =$$

$$[10(t-2) - 10e^{-1/3(t-2)}]u(t-2)$$



With Our Best Wishes

Automatic Control (1)

Course Staff

Thank You
For Your Attention

Mohamed Ahmed Ebrahim

Associate Prof. Dr . Mohamed Ahmed Ebrahim